JOURNAL OF COMPUTATIONAL PHYSICS 35, 426–430 (1980)

# Note

## The Effect of Nonzero $\nabla \cdot \mathbf{B}$ on the Numerical Solution of the Magnetohydrodynamic Equations*

It has been observed that even very small errors in satisfying the equation

$$\nabla \cdot \mathbf{B} = 0 \tag{1}$$

cause large errors in the solution of the magnetohydrodynamic equations when the equations are written in conservation form [1]. These errors are due to a formulation of the magnetic force in which numerical errors in the solution of Eq. (1) appear as a force parallel to the field. In this note, we show that the nonphysical parallel force is effectively eliminated, even when $\nabla \cdot \mathbf{B} \neq 0$, by writing the momentum equation in nonconservation form. The resulting formulation retains the conservation form of the induction and the energy equations.

In numerical calculations, $\nabla \cdot \mathbf{B}$ is typically small, but not zero. Nevertheless, difference equations are usually written to approximate magnetohydrodynamic equations in conservation form [1, 2], a form which is only correct when $\nabla \cdot \mathbf{B} = 0$. While $\nabla \cdot \mathbf{B} = 0$ initially, errors in the difference equations cause $\nabla \cdot \mathbf{B}$ to evolve as given by

$$\frac{\partial}{\partial t}(\nabla \cdot \mathbf{B}) = 0 + O(\Delta x^m, \Delta t^n), \tag{2}$$

where $m, n \geqslant 1$. When $\nabla \cdot \mathbf{B} \neq 0$, the magnetohydrodynamic equations are not in conservation form. The statement of magnetic flux conservation [3] and the momentum equation are written

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{u} \times \mathbf{B}) + \mathbf{u}(\nabla \cdot \mathbf{B}) = 0, \tag{3}$$

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p - \mathbf{B} \times (\nabla \times \mathbf{B}), \tag{4}$$

where $\mathbf{B}$ is the magnetic field intensity, $\mathbf{u}$ the fluid velocity, $\rho$ the mass density, and $p$ the fluid pressure. Neither of these equations is in conservation form. Likewise, when Eqs. (3) and (4) are combined with the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{5}$$





0021-9991/80/060426–05$02.00/0

to evaluate the energy integral

$$\frac{d}{dt}\int dV \left\{\frac{1}{2}\rho(\mathbf{u}\cdot\mathbf{u}) + \frac{1}{2}(\mathbf{B}\cdot\mathbf{B})/\mu\right\}$$

$$= \int dV \left\{\nabla\cdot\left[-\left(p+\frac{1}{2}\mathbf{B}\cdot\mathbf{B}/\mu\right)\mathbf{u} + \mathbf{B}(\mathbf{B}\cdot\mathbf{u})\right.\right.$$

$$-(\mathbf{B}\cdot\mathbf{u})\,\nabla\cdot\mathbf{B}$$

one finds the energy integral is not in conservation form. As a consequence, when $\nabla\cdot\mathbf{B}\neq 0$, the magnetic flux, momentum, and energy of an isolated system are not constants of the motion. When $\nabla\cdot\mathbf{B}=0$, Eq. (3) is obviously in conservation form, and the magnetic force in Eq. (4) can be written as the divergence of the Maxwell stress tensor

$$\mathbf{F} = \nabla\cdot\{-\tfrac{1}{2}(\mathbf{B}\cdot\mathbf{B})/\mathbf{I} + \mathbf{BB}\} \tag{7}$$

where **I** is the unit matrix.

When these equations are differenced, magnetic flux, energy, and momentum are constants of the motion even when $\nabla\cdot\mathbf{B}\neq 0$.

Since $\nabla\cdot B = O(\Delta x^m, \Delta t^n)$, both difference formulations are consistent with the differential equations with $\nabla\cdot\mathbf{B}=0$, which are the ones we wish to solve, but it would seem that the conservation form is preferable. However, exact conservation form may be imperative only for high-speed flows where Rankine–Hugoniot conditions across shocks would not be satisfied otherwise. In problems which are characterized by low-speed flow so that the equations must be integrated for many signal transit times, as in instability calculations for low-beta plasmas or calculations of equilibria, the conservation form of the momentum equation may be inappropriate. The difficulty is that the projection of the magnetic force given by Eq. (7) onto the magnetic field is not zero when $\nabla\cdot\mathbf{B}\neq 0$. Rather, it is proportional to $\nabla\cdot\mathbf{B}$ and is given by [4]

$$\mathbf{F}\cdot\mathbf{B} = (\mathbf{B}\cdot\mathbf{B})\,\nabla\cdot\mathbf{B}. \tag{8}$$

Since $\nabla\cdot B$ itself is a numerical error proportional to **B**, the error in the force relative to the fluid velocity will be the greater the lesser the fluid velocity is relative to the Alfven speed. Furthermore, in a numerical equilibrium solution $\mathbf{B}\cdot\nabla p$ will not be zero unless $\nabla\cdot\mathbf{B}=0$. In some instances, the variation in $\nabla\cdot\mathbf{B}$ may be such that no numerical equilibrium exists.

Just what can happen in a numerical equilibrium calculation with the momentum equation in conservation form is illustrated in Fig. 1. The evolution of a uniform plasma in a uniform magnetic field has been represented on a two-dimensional, initially rectilinear, Lagrangian computation mesh. As described in several earlier papers [4, 5], the method of solution is implicit, and mass, momentum, magnetic flux,

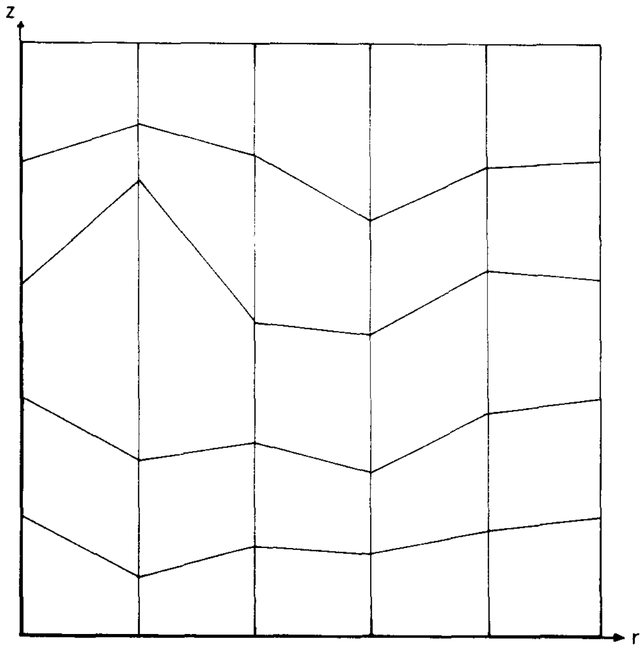


FIG. 1. The computation mesh for a Lagrangian calculation without correction for $\nabla \cdot \mathbf{B} \neq 0$ is shown after 209 time steps, corresponding to 160 signal transit times across the mesh. The axis of symmetry coincides with the left boundary; the velocity, $\mathbf{u}$, is zero on the top, right, and bottom boundaries. The initial field is parallel to the axis and uniform. The plasma beta is $1.3 \times 10^{-3}$. The displacements of the vertices of the mesh are parallel to the magnetic field, and result from velocities approximately equal to $2 \times 10^{-3}$ times the Alfven speed.

and energy are conserved for all values of $\nabla \cdot \mathbf{B}$. Although the formulation is Lagrangian, it is equivalent to the Eulerian conservation form outlined above.

After 160 signal transit times, flow velocities equal to $10^{-2}$ times the Alfven speed have developed aligned with the magnetic field and have deformed the Lagrangian mesh as shown in Fig. 1. This flow has no physical cause, and it eventually inverts cells and terminates the calculation. Yet, the flow corresponds to rather small errors relative to $\mathbf{B}$ in the solution of the equation $\nabla \cdot \mathbf{B} = 0$. In this case, the attempt to impose momentum conservation on a numerical solution when $\nabla \cdot \mathbf{B}$ is not zero caused significant and unacceptable derivation from physical behavior.

To eliminate the parallel force, one can formulate the magnetohydrodynamic equations in terms of fluxes for which $\nabla \cdot \mathbf{B} = 0$ automatically. One can also solve for a potential, $\phi$, from the equation

$$\nabla^2 \phi + \nabla \cdot \mathbf{B} = 0.$$

and define a new magnetic field, $\mathbf{B}' \equiv \mathbf{B} + \nabla \phi$ for which $\nabla \cdot \mathbf{B}' = 0$. However, one can eliminate the parallel force without having to solve a potential problem by differencing the nonconservative form of the momentum equation, Eq. (4). This equation is consistent with energy conservation if we write the induction equation in conservation form,

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{u} \times \mathbf{B}) = 0. \tag{9}$$

When Eqs. (4) and (9) are combined, the energy integral may be written

$$\frac{d}{dt} \int dV \left\{ \frac{1}{2} \rho(\mathbf{u} \cdot \mathbf{u}) + \frac{1}{2} (\mathbf{B} \cdot \mathbf{B})/\mu \right\} = \int dV \, \nabla \cdot \left[ - \left( p + \frac{1}{2} \mathbf{B} \cdot \mathbf{B}/\mu \right) + \mathbf{B}(\mathbf{B} \cdot \mathbf{u}) \right] \tag{10}$$

This equation, together with Eqs. (4), (5), and (9), comprise a system for which energy and magnetic flux are constants of the motion, momentum is conserved to $O(\Delta x^m, \Delta t^n)$, and $\mathbf{F} \cdot \mathbf{B}$ is zero for all $\nabla \cdot \mathbf{B}$.

The results of a numerical calculation in which the uniform plasma case has been repeated with the nonconservation form of the momentum equation are shown in

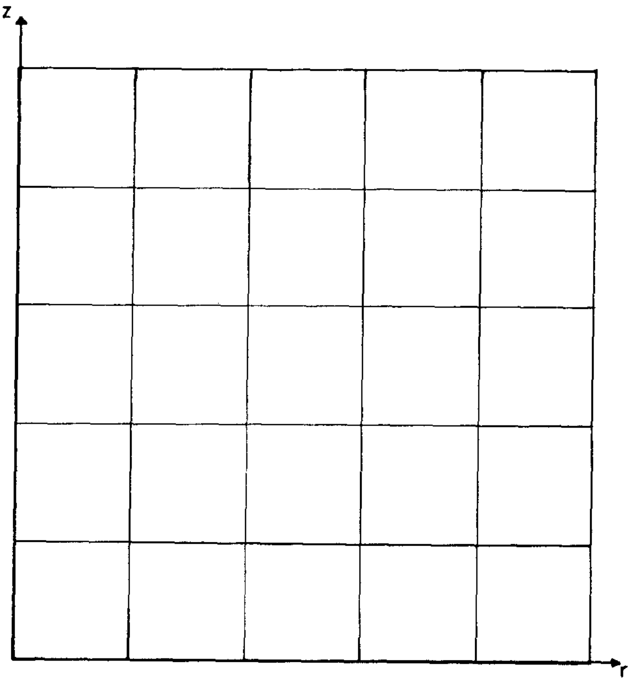


FIG. 2. The computation mesh for a Lagrangian calculation with corrections for $\nabla \cdot \mathbf{B} \neq 0$ is shown after 171 time steps, corresponding to 300 signal transit times across the mesh. The initial and boundary conditions are identical to those for the calculation shown in Fig. 1. No distortion of the mesh is evident, indicating that all velocities are very small.

Fig. 2. After 300 signal transit times across the mesh, the mesh appears undistorted. Small random velocities have developed, but the kinetic energy is only 1/4000 as much as it is with the conservative formulation and remaining constant. In this case, reformulation of the momentum equation eliminated the nonphysical behavior and brought the numerical solution much closer to the correct one.

Some other results using the conservation form of the momentum equation suggest a similar problem to the one we have encountered. In an explicit calculation [2] using leapfrog and Lax–Wendroff time differencing, the residual kinetic energy was decreased by $10^3$ when the $\nabla \cdot \mathbf{B}$ error was corrected. This suggests that reformulating the momentum equation in a form which eliminates the parallel magnetic force is of universal importance in computational magnetohydrodynamics.

## References

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Received: October 31, 1978; revised: May 22, 1979

J. U. Brackbill and D. C. Barnes

*Los Alamos Scientific Laboratory*
*University of California*
*Los Alamos, New Mexico 87545*